# $J_i$ , $K_i$ as Differential Operators Corresponding to Decomposed Chain SL (2,ℂ) ⊃$E_2$⊃ SO(2)

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## ABSTRACT

This leads to the fact that for the non-compact linear group SL(2,ℂ) (homomorphic to the Lorentz group), have the different possible chains of decompositions, has one more maximal subgroup$E_2$ which is the 2 dimensional Euclidean group generated by

$$L_1 = K_1 + J_2 \quad , \quad L_2 = J_1 - K_2 \quad , \quad J_3$$

and we are able to obtain in a quite simple manner $J_i$ , $K_i$ as differential operators with respect to the parameters which parameterize $E_2$corresponding to this chain.

SL (2,ℂ)⊃ $E_2$⊃SO(2),

**Keywords:** Differential Operator, Generators, SL (2,C), Unitary Irreducible Representation.

## INTRODUCTION

The subject of Representation Theory of continuous groups, developed a lot after the second world war, due to its important application in Nuclear Physics and Elementary Particle Physics. The first groups to be considered in this context, were the compact rotation groups SO (3) and S0 (4) and the non-compact rotation, i.e. Lorentz groups SO (2,1) and SO (3,1) which are homomorphic to the non compact linear group SU (1,1) and SL (2,C) respectively. A large number of papers on the Representation Theories of these group, were published [1-9].

An important aspect of Representation Theory is the problem of representing generators of Lie Algebra of the group, as differential operators. A number of open problems in this field, with potential use in Loop Quantum Gravity, exist. We have chosen such a problem and able to solve it and get results which will extend the theory of Spin Foam Models [8] and other parts of Loop Quantum Gravity.

## OBJECTIVE

We are concerned with the special linear groups SL(2,ℂ), which is homomorphic to the Lorentz group SO(3,1)which has 6 generators

$J_1$, $J_2$, $J_3$, $K_1$, $K_2$, $K_3$.

One of the prominent set of unitary irreducible representation (UIR) of this group, viz. continuous representations of the principal series, and are usually denoted by $D^{\rho n}$,their realization according to the chain of decomposition

$$\text{SL}(2,\mathbb{C}) \supset \text{SU}(2) \supset \text{SO}(2), \qquad (1.1)$$

$$\text{SL}(2,\mathbb{C}) \supset \text{SU}(1,1) \supset \text{SO}(2), \qquad (1.2)$$

$$\text{SL}(2,\mathbb{C}) \supset \text{SU}(1,1) \supset \text{SO}(1,1), \qquad (1.3)$$

$$\text{SL}(2,\mathbb{C}) \supset \text{SU}(1,1) \supset T_1 \qquad (1.4)$$

However, as far as expressions for generators as differential operators, are concerned, they were evaluated in the sixties, only for the chain (1.1). Those for the chains (1.2), (1.3) were evaluated in 2011 by Conrady and Hnybida [8] for use in spin-form model of Loop Quantum Gravity. In the present work, we may go beyond the maximal subgroup SU (2) and SU(1,1) of SL(2,C), and consider the third maximal subgroup $E_2$ of it, which is the 2 dimensional Euclidean group generated by

$$L_1 = K_1 + J_2 \quad , \quad L_2 = J_1 - K_2 \quad , \quad J_3 \qquad (1.5)$$

and we are able to obtain $J_i$ $K_i$ as differential operators corresponding to this chain.

$$SL(2,\mathbb{C}) \supset E_2 \supset SO(2), \qquad (1.6)$$

## METHODOLOGY

Several techniques for finding the generators of infinitesimal transformations, as differential operators, can be found in literature. As our problem is similar to those of Conrady and Hnybida, we adopted the technique using UIR matrix elements of infinitesimal elements of parent group, is given by Conrady and Hnybida [8], who use it for obtaining the differential operators for the chains of decomposition (1.2) and (1.3) of SL(2, C) .

In this paper we have started by showing that the set of functions $f$: $E_2 \rightarrow \mathbb{C}$, satisfies the covariance condition

$$f(\mathrm{U_z}(\theta)\mathrm{w}) = \mathrm{e}^{-\mathrm{in}\frac{\theta}{2}} f(\mathrm{w})$$

are in 1-1 correspondence with the set of functions $F(z_1,z_2)$ satisfies the homogeneity conditions

$$F(\alpha z_1,\alpha z_2) = \alpha^{-1+i\rho/2}\,(\alpha^{\times})^{-1+\frac{i\rho}{2}-\frac{n}{2}} F(z_1,z_2)$$

Going over to parametrization of group element $w$ of $E_2$ is

$$r(\varphi) r \hat{l}(-r) r\,(\psi)$$

After solving we get

$$w_1 = e^{\frac{i(\varphi+\psi)}{2}},$$

$$w_2 = re^{\frac{i(\varphi-\psi)}{2}}$$

By using the counter part of equation 56 of Conrady [8]

As we know the action of the representative of an arbitrary g∈ SL(2, ℂ ) on elements of carrier space, the action of (the representatives of) generators of infinitesimal transformations Ji, $K_i$ on these elements, are easily obtained.

## RESULT

We are able to obtain in a quite simple manner $J_i$ ,$K_i$ as differential operators.

The results obtained by Conrady and Hnybida have been used by a large number of people working on Spin Foam Models of Loop Quantum Gravity [ 4-7]. We hope that our results will enable these authors to extend their theories to an additional case not considered earlier. Thus, the result will not only fill an existing gap in the Mathematical theory, but will also have useful applications in Loop Quantum Gravity, a rapidly growing branch of Physics.

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