A Generalized Jensen Inequality for Concave Utilities: Theory and Economic Applications

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ABSTRACT

Making decision under uncertainty is a fundamental problem of economics and finance. Expected utility theory and mean variance portfolio analysis formalized the decision making under uncertainty. Jensen's inequality is widely used in expected utility theory, and nu- merous refinements have been proposed to make it more efficient and useful in economics and finance. This work presents a refinement of Jensen inequality, provided a generalized Jensen inequality of com- position $\varphi \circ U(x)$, where $U(x)$ is a concave utility function and φ a convex, non increasing mapping. Two alternative proofs are given: the first uses convexity of composition and the second uses subgradi- ent methods. The result is applied to the quadratic utility function for verification as an example. The generalized Jensen inequality for concave utilities strengthens the connection between convex analysis, economics, and finance.

Keywords: Jensen Inequality, Concave Utility, Convex Analysis, Expected Utility, Risk Aversion.

INTRODUCTION

Decision making under uncertain circumestances is core central to economics and finance. Expected utility theory was introduced by Bernoulli [1] nor- malizes the uncertainty by mathematical modeling. Von Neumann and Mor- genstern [2] formalized the framework, give the basis of behiovor modeling in risk. In utility theory, concave utility functions presents risk averse pref- erences and the individual prefer risk averse preferences is said to be risk averse. The fundamental result of convex analysis, Jensen inequality [3] play a vital role in expected utility. For a convex function f and real numbers

$x_1, ..., x_n$ with non-negative weights $p_1, ..., p_n$ such that $\sum_{i=1}^{n} p_i = 1$,

$$\left(\sum_{i=1}^{n} p_i x_i \right) \leq \sum_{i=1}^{n} p_i f(x_i)$$

with the inequality reversed if f is concave [3]. According to Jensen inequal- ity, for a concave utility function, the utility of certain wealth is greater than or equal to the expected utility of uncertain wealth. for an integrable random variable X,

$$f(\mathrm{E}[X]) \leq \mathrm{E}[f(X)],$$

again reversed for concave f [4, 5, 6].If a utility function is concave then

$$\mathrm{E}[U(X)] \leq U(\mathrm{E}U(X)),$$

which implies that the greater magnitude of the function, the higher degree of risk aversion [7, 8].

Over the decades numerous refinement have been made to Jensen Inequal- ity, to increase its application domain. Díaz-Barrero *et al.* improved Ky Fan-type inequalities by expanding them to convex combinations of means [9]. Persson and Nikolova identified the gap between f (E[X] and U (EU (X)) which is known as Jensen gap, and introduced the reverse Jensen inequal- ity [10]. Merhav [11] proposed Jensen-type inequalities, extending Jensen inequality to more complex forms such as convex compositions. These refine- ments generally assume the convex-convex or concave-concave compositions, where both functions share the same curvature.

Conversely, This work presents a generalized Jensen inequality where the outer function φ is convex, non increasing and the inner function U is concave. This result relaxes the restriction of the functions having the same curvature. The Quadratic utility function $U(x) = x - bx^2$ is a concave utility func-

tions which has a direct application in mean-variance portfolio analysis and asset allocation models [12, 13]. Along with its limitations such as negativity of marginal utility at some levels of wealth which violates non-satiation, the quadratic utility is more analytically tractable than other utility functions. Quadratic utility is analytically tractable despite these theoretical disadvan- tages, which are used as a benchmark in the recent robust and behavioral portfolio models [14, 15]. The recent research also proved its application in behavioral contexts and with model uncertainty [16], which makes its appli- cability to our framework an apparent choice.

OBJECTIVES

The aim is to establish a generalized Jensen inequality for concave utilities, specially for compositions of convex-concave functions, in order to strengthen the connection between convex analysis, risk aversion, and mean-variance portfolio anaysis.

METHODOLOGY

Two alternative proofs of generalized Jensen inequality for utilities are pro- vided, first by using the convexity of composition and second by using subgra- dient methods. The verification of result is applied to the quadratic utility function.

KEY FINDINGS

The generalized Jensen inequality for concave utilities, presented, is an exten- sion of classical Jensen inequality for composition of mixed curvature. The classical Jensen inequality is obtained from the proposed inequality. Direct link of proposed inequality to mean variance optimization strengthen the connection between convex analysis, economics and finance.

CONCLUSION

The proposed inequality enhances the application of Jensen inequality in utility theory and establishes stronger connection between convex analysis and economics. It provides new insights into risk preferences modeling, and links with diversification models, offers a new theoretical rationale for con- cave utilities, particularly the quadratic utility $U(x) = x - bx^2$, in financial economics. Future research could extend the framework for higher moments, such as skewness and Kurtosis. Furthermore, Modifying the inequality for behavioral utility methods and market dynamics of portfolio models presents another research direction.

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